The Raised Cosine Pulse

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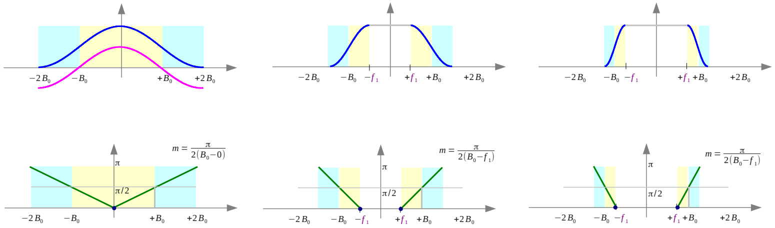
Physical Realization

- The Nyquist channel $P_{opt}(f)$: ideal
- the modified $P(f)$ decreases toward zero gradually rather than abruptly (a rectangle function)
- two parts
- Flat portion $0 \leq |f| \leq f_1$
- Roll-off portion $f_1 \leq |f| \leq 2B_0 - f_1$

Flat and Roll-off Portions

- one full cycle of the cosine function
- defined in the frequency domain
- raised up byan amount equal to its amplitude

- $P(f) = \frac{\sqrt{E}}{2B_0}$ $\quad (0 \leq |f| \leq f_1)$
- $P(f) = \frac{\sqrt{E}}{4B_0}\left\{1 + cos\left[\frac{\pi(|f| - f_1)}{2(B_0 - f_1)}\right]\right\}$ $\quad (f_1 \leq |f| \leq 2B_0 - f_1)$
- $P(f) = 0$ $\quad (2B_0 - f_1 \leq |f|)$



Raised Cosine Pulse Spectrum

- $P(f) = \frac{\sqrt{E}}{2B_0}$ $\quad (0 \le |f| \le f_1)$
- $P(f) = \frac{\sqrt{E}}{4B_0}\left\{1 + cos\left[\frac{\pi(|f| - f_1)}{2(B_0 - f_1)}\right]\right\}$ $\quad (f_1 \le |f| \le 2B_0 - f_1)$
- $P(f) = 0$ $\quad (2B_0 - f_1 \le |f|)$

- slope $\quad m = \frac{\pi}{2(B_0 - f_1)}$
- x intercept point $\quad (f_{1,}0) \quad x \Longrightarrow (x - f_1)$
- argument equation $\quad \theta = \frac{\pi(f - f_1)}{2(B_0 - f_1)}$
- raised cosine $\quad \frac{1}{2}\left\{1 + cos\left[\frac{\pi(|f| - f_1)}{2(B_0 - f_1)}\right]\right\}$

Roll-off Factor α

- $P(f) = \frac{\sqrt{E}}{2B_0}$ $\quad (0 \le |f| \le f_1)$
- $P(f) = \frac{\sqrt{E}}{4B_0}\left\{1 + cos\left[\frac{\pi(|f|-f_1)}{2(B_0-f_1)}\right]\right\}$ $\quad (f_1 \le |f| \le 2B_0 - f_1)$
- $P(f) = 0$ $\quad (2B_0 - f_1 \le |f|)$

- roll-off factor $\alpha = \frac{(B_0-f_1)}{B_0} = 1 - \frac{f_1}{B_0}$
- normalized by $\frac{2B_0}{\sqrt{E}}$
- normalized frequency $\frac{f}{B_0}$

$$p(t) = \sqrt{E}sinc(2B_0t)\left(\frac{cos(2\pi\alpha B_0t)}{1-16\alpha^2B_0^2t^2}\right)$$

Raised Cosine Pulse Spectrum & Shape

Raised Cosine Pulse Spectrum

- $P(f) = \frac{\sqrt{E}}{2B_0}$ $\qquad (0 \le |f| \le f_1)$
- $P(f) = \frac{\sqrt{E}}{4B_0}\left\{1 + cos\left[\frac{\pi(|f| - f_1)}{2(B_0 - f_1)}\right]\right\}$ $\qquad (f_1 \le |f| \le 2B_0 - f_1)$
- $P(f) = 0$ $\qquad (2B_0 - f_1 \le |f|)$

Raised Cosine Pulse Shape

- $p(t) = \sqrt{E} sinc(2B_0 t)\left(\frac{cos(2\pi\alpha B_0 t)}{1 - 16\alpha^2 B_0^2 t^2}\right)$

Raised Cosine Pulse Shape

- $p(t) = \sqrt{E} sinc(2B_0 t) \left(\frac{cos(2\pi\alpha B_0 t)}{1 - 16\alpha^2 B_0^2 t^2} \right)$
- $\sqrt{E} sinc(2B_0 t)$ Nyquist channel
  - makes zero crossings at the sampling instants $t = iT_b$
- $\left(\frac{cos(2\pi\alpha B_0 t)}{1 - 16\alpha^2 B_0^2 t^2} \right)$ decreases as $\frac{1}{|t|^2}$ for large $|t|$
  - reduces the tails of the pulse signficantly low
  - makes the transmitted signal insensitive to sampling time errors
  - the ISI error due to a timing error Δt decreases as $\alpha \to 1$

Raised Cosine Pulse Shape ($\alpha \to 1$)

- $p(t) = \sqrt{E} sinc(2B_0 t) \left(\frac{cos(2\pi\alpha B_0 t)}{1-16\alpha^2 B_0^2 t^2} \right)$
- the ISI error due to a timing error Δt decreases as $\alpha \to 1$
- $p(t) = \sqrt{E} \left(\frac{sin(2\pi B_0 t)}{2\pi B_0 t} \right) \left(\frac{cos(2\pi B_0 t)}{1-16B_0^2 t^2} \right)$

$$= \sqrt{E} \left(\frac{sin(4\pi B_0 t)}{2 \cdot 2\pi B_0 t} \right) \left(\frac{1}{1-16B_0^2 t^2} \right) = \sqrt{E} \left(\frac{sinc(4B_0 t)}{1-16B_0^2 t^2} \right)$$

Zero Crossings of Raised Cosine Pulse Shape ($\alpha \to 1$)

- $p(t) = \sqrt{E} sinc(2B_0 t) \left(\frac{cos(2\pi\alpha B_0 t)}{1 - 16\alpha^2 B_0^2 t^2} \right) \Rightarrow \sqrt{E} \left(\frac{sinc(4B_0 t)}{1 - 16 B_0^2 t^2} \right)$

- zero crossings of $sinc(4B_0 t)$: $t = k\frac{1}{4B_0} = k\frac{T_b}{2}$
- but, at $t = \pm\frac{T_b}{2} = \pm\frac{1}{4B_0}$,

  $\Longrightarrow 1 - 16B_0^2 t^2 = 0$ denominator is also zero

- $\frac{sin(4\pi B_0 t)}{4\pi B_0 t(1 - 16B_0^2 t^2)}$ when $t = \pm\frac{T_b}{2} = \pm\frac{1}{4B_0}$

  $\Longrightarrow \frac{4\pi B_0 cos(4\pi B_0 t)}{4\pi B_0 (1 - 16B_0^2 3t^2)} = \frac{1}{2}$

  $\Longrightarrow p(t) = 0.5\sqrt{E}$

- the same zero crossings: $t = \pm\frac{2}{2}T_b, \pm\frac{4}{2}T_b, \pm\frac{6}{2}T_b, \cdots$
- another zero crossings: $t = \pm\frac{3}{2}T_b, \pm\frac{5}{2}T_b, \pm\frac{7}{2}T_b, \cdots$

Transmission Bandwidth

- Transmission Bandwidth $B_T = 2B_0 - f_1$
- Roll-off factor $\alpha = \frac{(B_0 - f_1)}{B_0} = 1 - \frac{f_1}{B_0}$
- $B_T = B_0 + B_0 - f_1 = B_0 + \alpha B_0 = (1+\alpha)B_0$
- Excess Bandwidth $f_v = \alpha B_0$
- Roll-off factor = Excess bandwidth factor

When $\alpha \rightarrow 0$

- $f_v \rightarrow 0$
- $B_T \rightarrow B_0 = \frac{1}{2B_0}$ minimum bandwidth

When $\alpha \rightarrow 1$

- $f_v \rightarrow B_0$
- $B_T \rightarrow 2B_0 = \frac{1}{B_0}$ doubled bandwidth
- used for synchronizing the receiver to the transmitter

The Infinite Replicas of the Raised Cosine Pulse Spectrum

The Infinite Replication

The infinite summation of replicas of the raised cosine pulse spectrum, spaced by $2B_0$Hz, equals a constant.

$$\sum_{m=-\infty}^{\infty} P(f - m2B_0) = \frac{\sqrt{E}}{2B_0}$$

$$\sum_{n=-\infty}^{\infty} p(\tfrac{n}{2B_0})\delta(t - \tfrac{n}{2B_0}) \leftrightarrows 2B_0 \sum_{m=-\infty}^{\infty} P(f - m2B_0)$$

$$p(t) = \sqrt{E} sinc(2B_0 t) \left(\frac{cos(2\pi\alpha B_0 t)}{1-16\alpha^2 B_0^2 t^2} \right)$$

$$p(\tfrac{n}{2B_0}) = \sqrt{E} sinc(2B_0 \tfrac{n}{2B_0}) \left(\frac{cos(2\pi\alpha B_0 \frac{n}{2B_0})}{1-16\alpha^2 B_0^2 \left(\frac{n}{2B_0}\right)^2} \right) = \sqrt{E} sinc(n) \left(\frac{cos(\pi n\alpha)}{1-4n^2\alpha^2} \right)$$

$sinc(n) = \frac{sin(n\pi)}{n\pi}$ $(= 1$ when $n = 0$, $= 0$ when $n = \pm1, \pm2, \cdots)$
$cos(\pi n\alpha) = 1$when $n = 0$

The Infinite Replicas of the Raised Cosine Pulse Spectrum

$$p(\frac{n}{2B_0}) = \sqrt{E} sinc(n) \left(\frac{cos(\pi n \alpha)}{1-4n^2\alpha^2} \right)$$

$$sinc(n) = \frac{sin(n\pi)}{n\pi} \; (= 1 \text{ when } n = 0 \;, = 0 \text{ when } n = \pm 1, \pm 2, \cdots)$$

$$cos(\pi n \alpha) = 1 \text{when } n = 0$$

$$p(\frac{n}{2B_0}) = \sqrt{E} \text{ when } n = 0$$

$$= 0 \text{ when } n = 0$$

$$\sqrt{E}\delta(t) \leftrightarrows 2B_0 \sum_{m=-\infty}^{\infty} P(f - m2B_0)$$

$$\frac{\sqrt{E}}{2B_0}\delta(t) \leftrightarrows \sum_{m=-\infty}^{\infty} P(f - m2B_0)$$

The Criterion for Zero ISI

Given the modified pulse shape $p(t)$ for transmitting data over an imperfect channel using discrete pulse-amplitude modulation at the data rate $1/T$, the pulse shape $p(t)$ eleminates intersymbol interference if, and only if, its spectrum $P(f)$ satisfies the condition

$$\sum_{m=-\infty}^{\infty} P(f - m/T) = \sum_{m=-\infty}^{\infty} P(f - m2B_0) = const \qquad |f| \leq \frac{1}{2T}$$

Reference

[1] S. Haykin, M Moher, “Introduction to Analog and Digital Communications”, 2ed